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Bounds for Lebesgue Functions for Freud Weights

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Let $W(x) := e^{-Q(x)}$, $x \in \mathbb{R}$, where $Q(x)$ is even and continuous in $\mathbb{R}$, Q'' is continuous in $(0, \infty)$, and $Q' > 0$ in $(0, \infty)$, while for some $A, B > 1$,

$$A \leq \left[\frac{d}{dx}(xQ'(x))\right] \Big/ Q'(x) \leq B, \qquad x \in (0, \infty).$$

Let $p_n(W^2, x)$ denote the nth orthonormal polynomial for the weight $W^2(x)$, $x_{kn}(W^2)$ the kth zero of $p_n(W^2, x)$, and $l_{kn}(x)$ the fundamental polynomials. Moreover let a_n denote the nth *Mhaskar–Rahmanov–Saff* number for Q and let $\sigma \in (0, 1)$. Then we show that the nth weighted Lebesgue function satisfies uniformly for $|x| \leq \sigma a_n$,

$$W(x) \sum_{k=1}^{n} |l_{kn}(x)| W^{-1}(x_{kn})(1 + |x_{kn}|)^{-\alpha}$$

$$\sim (1 + |x|)^{-\alpha} + \sqrt{a_n} |p_n(W^2, x)| W(x) \left\{(1 + |x|)^{-\alpha} \log n + (1 + |x|)^{-\hat{\alpha}}\right\},$$

$$\leq C\left\{(1 + |x|)^{-\alpha} \log n + (1 + |x|)^{-\hat{\alpha}}\right\},$$

where $\alpha \geq 0$ and $\hat{\alpha} := \min\{1, \alpha\}$. We also modify this result to the whole real line.

1. INTRODUCTION AND RESULTS

We consider $W := e^{-Q}$, where $Q: \mathbb{R} \to \mathbb{R}$ is even and continuous in $\mathbb{R}$, $Q' > 0$ in $(0, \infty)$, Q'' is continuous in $(0, \infty)$, while for some $A, B > 1$,

$$A \leq \left[\frac{d}{dx}(xQ'(x))\right] \Big/ Q'(x) \leq B, \qquad x \in \mathbb{R}. \tag{1.1}$$

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We call such a W a *Freud Weight*. An archetypal example is

$$W_\beta := \exp(-|x|^\beta), \qquad \beta > 1.$$

Corresponding to W^2 is a sequence of orthonormal polynomials $\{p_n(x)\}$, where

$$p_n(x) := p_n(W^2, x) = \gamma_n x^n + \cdots,$$

is the nth orthonormal polynomial of W^2 and $\gamma_n > 0$ is its leading coefficient. The zeros of $p_n(x)$ will be denoted by

$$-\infty < x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} < +\infty$$

arranged in increasing order.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\lim_{|x|\to\infty} |f(x)|W(x)(1+|x|)^\alpha = 0, \tag{1.2}$$

for $\alpha \geq 0$.

We define the error of polynomial approximation to f from the space $\mathscr{P}_{n-1}$ of all polynomials of degree at most $n-1$ by

$$E_n(f) := \inf_{P\in\mathscr{P}_{n-1}} \left\|(f-P)(x)(1+|x|)^\alpha W(x)\right\|_{L_\infty(\mathbb{R})},$$

and so there exists a unique $P^* \in \mathscr{P}_{n-1}$ such that

$$E_n(f) = \left\|(f-P^*)(x)(1+|x|)^\alpha W(x)\right\|_{L_\infty(\mathbb{R})}$$

since $\mathscr{P}_{n-1}$ is finite dimensional (cf. [1, p. 108]).

Let $L_n[f] \in \mathscr{P}_{n-1}$ denote the Lagrange interpolation polynomial to f at the zeros of $p_n(x)$. Then

$$\begin{aligned}&|(f-L_n[f])(x)|W(x)\\ &\qquad \leq E_n(f)(1+|x|)^{-\alpha} + W(x)\sum_{k=1}^{n}|l_{kn}(x)|\,|(f-P^*)(x_{kn})|,\end{aligned}$$

where

$$l_{kn}(x) := \frac{p_n(x)}{p_n'(x_{kn})(x-x_{kn})}$$

are the fundamental polynomials associated with W^2. We then have

$$|(f - L_n[f])(x)W(x)| \le E_n(f)(1 + |x|)^{-\alpha} + E_n(f)W(x) \sum_{k=1}^{n} |l_{kn}(x)|W^{-1}(x_{kn})(1 + |x_{kn}|)^{-\alpha}. \tag{1.3}$$

We thus define the nth *Lebesgue function* associated with the rate of decay in (1.2) by

$$\Lambda_n(x) := W(x) \sum_{k=1}^{n} |l_{kn}(x)|W^{-1}(x_{kn})(1 + |x_{kn}|)^{-\alpha}. \tag{1.4}$$

Our objective in this paper is to determine the correct bounds for $\Lambda_n(x)$ on the whole real line. Freud in [2, 3] studied the Lebesgue functions associated with compactly supported distributions and the Hermite Weight $\exp(-x^2/2)$, respectively. Nevai on the other hand established bounds of Lebesgue functions for the Laguerre Weight in [7] and then generalized his work to cover Laguerre, Jacobi, and Hermite Weights in [8]. For more work on this subject the reader can also refer to Szabados and Vertesi [9] and Knopfmacher [4], wherein bounds on Lebesgue functions were given for a subclass of the weights considered in this paper.

This paper deals with the bounds of Lebesgue functions associated with the same class of Freud Weights as studied by Levin and Lubinsky in [5]. To state our result we need some notation:

(1) Throughout, $L, C, C_1, C_2, \dots$ are positive constants independent of n and $x \in \mathbb{R}$. The same symbol does not necessarily denote the same constant in different occurrences.

(2) We use $\sim$ notation in the following sense:

$$f(x) \sim g(x)$$

if there exist positive constants C_1 and C_2 such that for the relevant range of x,

$$C_1 \le \frac{f(x)}{g(x)} \le C_2.$$

(3) For $u > 0$, the uth *Mhaskar–Rahmanov–Saff* a_u is the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 a_u t Q'(a_u t)\, dt/\sqrt{1 - t^2}. \tag{1.5}$$

(4) For $\alpha \geq 0$,

$$\hat{\alpha} := \min\{1, \alpha\}, \tag{1.6}$$

$$\psi_n(x) := \max\left\{n^{-2/3}, 1 - \frac{|x|}{a_n}\right\}, \qquad n \geq 1, x \in \mathbb{R}, \tag{1.7}$$

and

$$\log^*(x) := \begin{cases} 1, & \alpha \neq 1 \\ \log(x), & \alpha = 1. \end{cases} \tag{1.8}$$

(5) We set

$$x_{0n} := x_{1n}(1 + n^{-2/3}) \qquad \text{and} \qquad x_{n+1,n} := x_{nn}(1 + n^{-2/3}). \tag{1.9}$$

Our main result is the following theorem. Recall that ψ_n, $\log^*$, and $\hat{\alpha}$ are defined by (1.6)–(1.8).

THEOREM 1.1. (a) *There exists n_0 and C_1, C_2 such that for $n \geq n_0$ and $|x| \leq 2a_n$,*

$$\begin{aligned} \Lambda_n(x) &\leq C_1(1 + |x|)^{-\alpha} \\ &\quad + C_1\sqrt{a_n}|p_nW|(x)\Big\{(1 + |x|)^{-\alpha}\psi_n(x)^{1/4}\log\big[2n^{2/3}\psi_n(x)\big] \\ &\quad + (1 + |x|)^{-\hat{\alpha}}\log^* n\Big\}. \qquad (1.10) \\ &\leq C_2(1 + |x|)^{-\alpha}\log\big[2n^{2/3}\psi_n(x)\big] + C_2 n^{1/6}(1 + |x|)^{-\hat{\alpha}}\log^* n. \qquad (1.11) \end{aligned}$$

(b) *Let $\sigma \in (0, 1)$. There exists n_0 and C_3 such that uniformly for $n \geq n_0$ and $|x| \leq \sigma a_n$,*

$$\Lambda_n(x) \sim (1 + |x|)^{-\alpha} + \sqrt{a_n}|p_nW|(x)\{(1 + |x|)^{-\alpha}\log n + (1 + |x|)^{-\hat{\alpha}}\} \tag{1.12}$$

$$\leq C_3(1 + |x|)^{-\hat{\alpha}} + C_3(1 + |x|)^{-\alpha}\log n. \tag{1.13}$$

(c) *There exists n_0 such that uniformly for $n \geq n_0$ and $|x| \geq 2a_n$,*

$$\Lambda_n(x) \sim \sqrt{a_n}|p_nW|(x)\left\{\frac{1}{|x|}a_n^{1-\hat{\alpha}}\log^* n\right\}. \tag{1.14}$$

Remarks. (I) Observe that we don't have $\sim$ for $\sigma a_n \leq |x| \leq 2a_n$. In fact our proof shows that if $k(x) = k(x, n)$ is such that $x_{k(x),n}$ is the closest zero of $p_n(x)$ to x, (and we define $x_{-2,n}, x_{-1,n}, x_{n+2,n}, x_{n+3,n}$ much as at (1.9)) then uniformly for $n \geq 1$ and $|x| \leq 2a_n$,

$$\Lambda_n(x) - W(x) \sum_{k \in [k(x)-3, k(x)+3]} |l_{kn}(x)| W^{-1}(x_{kn})(1 + |x_{kn}|)^{-\alpha}$$
$$\sim \sqrt{a_n} |p_n W|(x) \Big\{ (1 + |x|)^{-\alpha} \psi_n(x)^{1/4} \log\big[2n^{2/3}\psi_n(x)\big] + (1 + |x|)^{-\hat{\alpha}} \log^* n \Big\}. \quad (1.15)$$

It is only the "closest terms" for which we cannot provide a suitable lower bound.

(II) An interesting feature occurs for $|1 - |x|/a_n| \leq Cn^{-2/3}$. For this range $\psi_n(x) \sim n^{-2/3}$, and (1.11) becomes for $\alpha \neq 1$,

$$\Lambda_n(x) \leq C_4(1 + |x|)^{-\alpha} + C_4 n^{1/6}(1 + |x|)^{-\hat{\alpha}}$$

in view of known bounds on $|p_n W|(x)$ (see (2.7) and (2.8) below). So the characteristic factor of $\log n$ disappears for x close to a_n.

2. Preliminary Results

The proof of the main result is a consequence of a number of lemmas.

LEMMA 2.1. (a) *For* $n \geq 1$,

$$\left| \frac{x_{1n}}{a_n} - 1 \right| \leq Cn^{-2/3} \quad (2.1)$$

and uniformly for $n \geq 3$ *and* $1 \leq k \leq n$,

$$x_{k-1,n} - x_{k+1,n} \sim \frac{a_n}{n} \psi_n(x_{kn})^{-1/2}. \quad (2.2)$$

(b) *Uniformly for* $1 \leq k \leq n-1$ *and* $n \geq 2$,

$$|p_n W|(x_{kn}) \sim a_n^{-1/2} \psi_n(x_{kn})^{1/4}. \quad (2.3)$$

(c) $Q'(x)$ *is increasing in* $(0, \infty)$ *and given* $0 < \alpha < \beta < \infty$,

$$Q'(x) \sim \frac{n}{a_n}, \qquad \textit{uniformly for } x \in [\alpha a_n, \beta a_n]. \quad (2.4)$$

Proof. (a) This is Corollary 1.2(a) in [5].

(b) This is Corollary 1.3 in [5].

(c) This is Lemma 5.1(c) in [5]. ▮

LEMMA 2.2. (a) *Uniformly for* $n \geq 1$, $1 \leq k \leq n$, *and* $x \in \mathbb{R}$,

$$|l_{kn}(x)| \sim \frac{a_n^{3/2}}{n} W(x_{kn}) \psi_n(x_{kn})^{-1/4} |p_n(x)| / |x - x_{kn}|. \quad (2.5)$$

(b) *Uniformly for* $n \geq 1$, $1 \leq k \leq n$, *and* $x \in \mathbb{R}$,

$$|l_{kn}(x)| W^{-1}(x_{kn}) W(x) \leq C. \quad (2.6)$$

(c)

$$\text{(i)} \quad \sup_{x \in \mathbb{R}} |p_n W|(x) \left|1 - \frac{|x|}{a_n}\right|^{1/4} \sim a_n^{-1/2}. \quad (2.7)$$

$$\text{(ii)} \quad \sup_{x \in \mathbb{R}} |p_n W|(x) \sim n^{1/6} a_n^{-1/2}. \quad (2.8)$$

Proof. Parts (a) and (b) are Lemma 2.6 in [6]. Part (c) is Corollary 1.4 in [5]. ▮

We now turn to $\Lambda_n(x)$. Let $x_{k(x),n}$ denote the closest abscissa to x. We can assume $x \geq 0$. Now choose $\delta > 0$ small enough and $M > 0$ large enough such that

$$\{k : |k(x) - k| \leq 3\} \subset \left\{k : |x - x_{kn}| \leq M \frac{a_n}{n} \psi_n(x)^{-1/2}\right\}. \quad (2.9)$$

This is possible because of (2.2).

We then split $\Lambda_n(x)$ as follows: Let

$$\mathcal{S}_1 := \left\{k : |x - x_{kn}| \leq \delta \frac{a_n}{n} \psi_n(x)^{-1/2}\right\};$$

$$\mathcal{S}_2 := \left\{k : |x - x_{kn}| \in \frac{a_n}{n} \psi_n(x)^{-1/2}(\delta, M)\right\};$$

$$\mathcal{S}_3 := \left\{k : |x - x_{kn}| \geq M \frac{a_n}{n} \psi_n(x)^{-1/2}\right\}.$$

Then

$$\Lambda_n(x) = \left[\Sigma_{\mathscr{S}_1} + \Sigma_{\mathscr{S}_2} + \Sigma_{\mathscr{S}_3}\right] W(x)|l_{kn}(x)|W(x_{kn})^{-1}(1 + |x_{kn}|)^{-\alpha}$$
$$=: \Sigma_1(x) + \Sigma_2(x) + \Sigma_3(x).$$

Next we estimate each sum.

$\Sigma_1(x)$. Observe that because of the spacing (2.2), Σ_1 has a finite number of terms. Using (2.6) we obtain

$$\Sigma_1(x) \le C \sum_{k \in \mathscr{S}_1} (1 + |x_{kn}|)^{-\alpha}$$
$$\le C_1(1 + |x|)^{-\alpha} \tag{2.10}$$

which is an easy consequence of (2.2). From (2.2) it follows that for $2 \le k \le n - 1$,

$$1 + |t| \sim 1 + |x_{kn}|, \qquad t \in [x_{k+1,n}, x_{k-1,n}]. \tag{2.11}$$

Now it is known that if $x \in [x_{k+1,n}, x_{kn}]$, for some $1 \le k \le n - 1$, then (see [9, p. 76])

$$l_{k+1,n}(x) + l_{kn}(x) \ge 1.$$

Assume for simplicity that $k = k(x)$ (if not, then $x \in [x_{kn}, x_{k-1,n}]$ and the argument is similar). Then

$$W(x) \sum_{j=k(x)}^{k(x)+1} |l_{jn}(x)| W^{-1}(x_{jn})(1 + |x_{jn}|)^{-\alpha}$$
$$\ge C(1 + |x|)^{-\alpha} W(x) \min\left\{W^{-1}(x_{k(x),n}), W^{-1}(x_{k(x)+1,n})\right\} \sum_{j=k(x)}^{k(x)+1} |l_{jn}(x)|$$
$$\ge C(1 + |x|)^{-\alpha} W(x) \min\left\{W^{-1}(x_{k(x),n}), W^{-1}(x_{k(x)+1,n})\right\},$$

by the abovementioned inequality.

Now if $|x| \le \sigma a_n$, then for n large enough, the spacing (2.2) gives

$$x_{kn} - x_{k+1,n} \sim \frac{a_n}{n}$$

so that for $j = k, k+1$, and some ξ between x, x_{jn},

$$\begin{aligned} |Q(x_{jn}) - Q(x)| &= |Q'(\xi)|\,|x_{jn} - x| \\ &\le Q'(a_n) C_1 \frac{a_n}{n} \\ &\le C_2 \end{aligned}$$

by Lemma 2.1(c). Then for $j = k, k+1$,

$$W(x) W^{-1}(x_{jn}) = e^{Q(x_{jn}) - Q(x)} = e^{O(1)} \sim 1,$$

uniformly for $|x| \le \sigma a_n$. So the above inequality becomes

$$\begin{aligned} & W(x) \sum_{j=k(x)}^{k(x)+1} |l_{jn}(x)| W^{-1}(x_{jn}) (1 + |x_{jn}|)^{-\alpha} \\ & \qquad \ge C(1+|x|)^{-\alpha}, \qquad |x| \le \sigma a_n. \end{aligned} \tag{2.12}$$

In particular, if $\Sigma_1(x)$ contains the terms in the last sum, we obtain from (2.10) and (2.12) that

$$\Sigma_1(x) \sim (1 + |x|)^{-\alpha}, \qquad |x| \le \sigma a_n. \tag{2.13}$$

$\Sigma_2(x)$. Observe that for $M > 0$ large enough but fixed, the number of terms in the set

$$\mathscr{S}_2 := \left\{ k : |x - x_{kn}| \in \frac{a_n}{n} \psi_n(x)^{-1/2} (\delta, M) \right\}$$

is bounded independently of x and n. Also $\psi_n(x) \sim \psi_n(x_{kn})$ and $(n/a_n)\psi_n(x)^{1/2}|x - x_{kn}| \sim 1$. Using (2.5) we obtain, if the sum is non-empty,

$$\begin{aligned} \Sigma_2(x) &\sim \sqrt{a_n}\, |p_n W|(x) \sum_{k \in \mathscr{S}_2} \frac{a_n}{n} \frac{\psi_n(x_{kn})^{-1/4}}{|x - x_{kn}|(1 + |x_{kn}|)^{\alpha}} \\ &\sim \sqrt{a_n}\, |p_n W|(x) \sum_{k \in \mathscr{S}_2} \psi_n(x)^{1/4} (1 + |x|)^{-\alpha} \\ &\sim \sqrt{a_n}\, |p_n W|(x) \psi_n(x)^{1/4} (1 + |x|)^{-\alpha}. \end{aligned} \tag{2.14}$$

$\Sigma_3(x)$. Let

$$J_n := [x_{n+1,n}, x_{0n}] \setminus [x_{k(x)+3,n}, x_{k(x)-3,n}].$$

From (2.9) it follows that

$$[x_{n+1,n}, x_{0n}] \setminus \left(x - M\frac{a_n}{n}\psi_n(x)^{-1/2}, x + M\frac{a_n}{n}\psi_n(x)^{-1/2}\right) \subset J_n.$$

We then estimate

$$\hat{\Sigma}_3(x) := \sum_{x_{kn} \in J_n} |l_{kn}(x)| W^{-1}(x_{kn})(1 + |x_{kn}|)^{-\alpha}$$

instead. First note that from (2.5) and then (2.2),

$$\hat{\Sigma}_3(x) \sim \sqrt{a_n}\,|p_n W|(x) \sum_{k \notin [k(x)-3, k(x)+3]} \frac{a_n}{n} \frac{\psi_n(x_{kn})^{-1/4}}{|x - x_{kn}|(1 + |x_{kn}|)^{-\alpha}}$$

$$\sim \sqrt{a_n}\,|p_n W|(x) \sum_{k \notin [k(x)-3, k(x)+3]} \frac{(x_{k-1,n} - x_{k+1,n})\psi_n(x_{kn})^{1/4}}{|x - x_{kn}|(1 + |x_{kn}|)^{\alpha}}.$$

Now as $k \notin [k(x) - 3, k(x) + 3]$,

$$|x - x_{kn}| \sim |x - t|, \qquad t \in [x_{k+1,n}, x_{k-1,n}].$$

This follows from

$$\left|\frac{x - x_{kn}}{x - t}\right| \le 1 + \frac{x_{k-1,n} - x_{k+1,n}}{x - x_{kn}} \le C.$$

The lower bound is obtained in a similar way. Thus we have

$$\hat{\Sigma}_3(x) \sim \sqrt{a_n}\,|p_n W|(x) \int_{J_n} \psi_n(t)^{1/4}(1 + |t|)^{-\alpha}/|x - t|\,dt, \quad (2.15)$$

where J_n is as defined earlier and we set

$$x_{n+3,n} = x_{n+2,n} = x_{n+1,n} \qquad \text{and} \qquad x_{-2,n} = x_{-1,n} = x_{0n}.$$

We have also used (2.11) and similar relations for $|x - t|$.

Now we turn to the estimation of

$$I := \int_{J_n} \psi_n(t)^{1/4}(1 + |t|)^{-\alpha}/|x - t|\,dt. \quad (2.16)$$

We consider 6 ranges of x.

LEMMA 2.3. *Let $x \in [0, 2]$. Then for $n \geq n_0$, n_0 large enough and independent of x,*

$$I \sim \log n.$$

Proof.

$$\begin{aligned} I &\sim \int_{J_n \cap [-4,4]} 1/|x-t|\,dt + \int_4^{a_n} \psi_n(t)^{1/4} t^{-\alpha-1}\,dt \\ &\sim \log n + \begin{cases} \log n, & \alpha = 0 \\ 1, & \alpha > 0 \end{cases} \\ &\sim \log n. \end{aligned}$$

∎

LEMMA 2.4. *Let $x \in [2, (3/4)a_n]$. Then*

$$I \sim (1+|x|)^{-\alpha} \log n + (1+|x|)^{-\hat{\alpha}}.$$

Proof. Now for $t \in \{|t| \geq (7/8)a_n\} \cap J_n$,

$$1 + |t| \sim |t| \qquad \text{and} \qquad |x - t| \sim |t|,$$

and so

$$\begin{aligned} I_1 &:= \int_{\{|t| \geq (7/8)a_n\} \cap J_n} \psi_n(t)^{1/4} (1+|t|)^{-\alpha} / |x-t|\,dt \\ &\sim \int_{[(7/8)a_n, (8/9)a_n]} t^{-\alpha-1}\,dt \\ &\sim a_n^{-\alpha}. \end{aligned}$$

Furthermore,

$$\begin{aligned} I_2 &:= \int_{\{|t| \leq (7/8)a_n\} \cap J_n} \psi_n(x)^{1/4} (1+|t|)^{-\alpha} / |x-t|\,dt \\ &\sim \int_{\{|t| \leq (7/8)a_n\} \cap J_n} (1+|t|)^{-\alpha} / |x-t|\,dt, \end{aligned}$$

since $\psi_n(t) \sim 1$ for this range of t.

Observe that

$$\{|t| \leq (7/8)a_n\} \cap J_n = \{|t| \leq (7/8)a_n\} \setminus [x_{k(x)+3,n}, x_{k(x)-3,n}].$$

Now we split this set into $J^{(l)}$, for $l = 1, 2, 3$, where

$$t \in J^{(1)} \Rightarrow |t| \leq |x|/2,$$

and thus

$$|x-t| \sim |x|.$$

$$t \in J^{(2)} \Rightarrow |t| \geq \frac{8}{7}|x|,$$

and thus

$$|x-t| \sim |t|$$

and

$$t \in J^{(3)} \Rightarrow \frac{1}{2}|x| < |t| < \frac{8}{7}|x|,$$

and thus

$$|t| \sim |x|.$$

First,

$$\begin{aligned} I_{21} &:= \int_{J^{(1)}} (1+|t|)^{-\alpha}/|x-t|\, dt \\ &\sim \frac{1}{|x|}\int_{J^{(1)}} (1+|t|)^{-\alpha}\, dt \\ &\sim \frac{1}{|x|}\int_0^{|x|/2} (1+t)^{-\alpha}\, dt \\ &\sim |x|^{-\hat{\alpha}} \log^*|x|. \end{aligned}$$

(Recall the definitions (1.6) and (1.8) of $\hat{\alpha}$ and $\log^* n$.)

$$\begin{aligned} I_{22} &:= \int_{J^{(2)}} (1+|t|)^{-\alpha}/|x-t|\, dt \sim \int_{J^{(2)}} t^{-\alpha-1}\, dt \\ &\sim \int_{(8/7)|x|}^{a_n} t^{-\alpha-1}\, dt \sim \begin{cases} x^{-\alpha}, & \alpha > 0 \\ \log(a_n/2x), & \alpha = 0 \end{cases} \\ &\leq C_3 \begin{cases} (1+|x|)^{-\alpha}, & \alpha > 0 \\ \log n, & \alpha = 0 \end{cases} \\ &\leq C_4 (1+|x|)^{-\alpha} \log n. \end{aligned}$$

Next

$$
\begin{aligned}
I_{23} &:= \int_{J(3)} (1+|t|)^{-\alpha} / |x-t|\, dt \\
&\sim |x|^{-\alpha} \int_{J(3)} 1/|x-t|\, dt \\
&\sim |x|^{-\alpha} \int_{[x/2,2x]\setminus[x_{k(x)+3,n},\, x_{k(x)-3,n}]} 1/|x-t|\, dt \\
&\sim |x|^{-\alpha} \int_{[1/2,2]\setminus[x_{k(x)+3,n}/x,\, x_{k(x)-3,n}/x]} 1/|1-s|\, ds \\
&\sim |x|^{-\alpha} \left\{ \log\left(\frac{x/2}{x - x_{k(x)+3,n}} \right) + \log\left(\frac{x}{x_{k(x)-3,n} - x} \right) \right\} \\
&\sim |x|^{-\alpha} \log n
\end{aligned}
$$

as

$$
\left| 1 - \frac{x_{k(x)+3,n}}{x} \right| = \left| \frac{x - x_{k(x)+3,n}}{x} \right| \begin{cases} \leq C_4 a_n / n \\ \geq C_5 / n \end{cases}
$$

since $x_{k(x)-3,n} - x_{k(x)+3,n} \sim a_n/n$, for $x \in [2, (3/4)a_n]$.

So

$$
\begin{aligned}
I &= I_1 + I_2 \\
&\sim a_n^{-\alpha} + I_{21} + I_{22} + I_{23} \\
&\sim a_n^{-\alpha} + (1+|x|)^{-\hat{\alpha}} \log^* |x| + O\big((1+|x|)^{-\alpha} \log n\big) \\
&\quad + (1+|x|)^{-\alpha} \log n \\
&\sim \begin{cases} (1+|x|)^{-\hat{\alpha}} + (1+|x|)^{-\alpha} \log n, & \alpha \neq 1 \\ (1+|x|)^{-\alpha} \log n, & \alpha = 1 \end{cases} \\
&\sim (1+|x|)^{-\alpha} \log n + (1+|x|)^{-\hat{\alpha}}. \quad \blacksquare
\end{aligned}
$$

Now Lemmas 2.3 and 2.4 and Eqs. (2.15) and (2.16) yield;

LEMMA 2.5. *For $x \in [0, (3/4)a_n]$,*

$$
\hat{\Sigma}_3(x) \sim \sqrt{a_n}\, |p_n W|(x) \left\{ (1+|x|)^{-\alpha} \log n + (1+|x|)^{-\hat{\alpha}} \right\}.
$$

LEMMA 2.6. *Let* $x \in [(3/4)a_n, a_n(1 - Ln^{-2/3})]$, *where* $L > 0$ *is so large that* $x_{3n} \geq a_n(1 - Ln^{-2/3})$. *Then*

$$I \sim a_n^{-\hat{\alpha}} \log^* n + a_n^{-\alpha} \psi_n(x)^{1/4} \log\left[2n^{2/3}\psi_n(x)\right].$$

Proof. In this case we have

$$1 - (x/a_n) \leq 1/4 \qquad \text{and} \qquad 1 - (x/a_n) \geq Ln^{-2/3}.$$

Here

$$I_3 := \int_{(-\infty, a_n/2) \cap J_n} \psi_n(t)^{1/4}(1 + |t|)^{-\alpha}/|x - t|\, dt$$

$$\sim a_n^{-1} \int_{(-\infty, a_n/2) \cap J_n} \psi_n(t)^{1/4}(1 + |t|)^{-\alpha}\, dt$$

since $|x - t| \sim a_n$.

Thus

$$I_3 \sim \left[a^{-1} \int_{[x_{n+1,n}, -a_n/2] \setminus [x_{k(x)+3,n}, x_{k(x)-3,n}]} \psi_n(t)^{1/4}(1 + |t|)^{-\alpha}\, dt \right.$$

$$\left. + a^{-1} \int_{[-a_n/2, a_n/2]} (1 + |t|)^{-\alpha}\, dt \right]$$

$$:= I_{31} + I_{32}.$$

Now

$$I_{31} \leq C_6 a_n^{-\hat{\alpha}} \log^* n \qquad \text{and} \qquad I_{32} \sim a_n^{-\hat{\alpha}} \log^* n.$$

Therefore

$$I_3 \sim a_n^{-\hat{\alpha}} \log^* n.$$

Next we deal with

$$I_4 := \int_{[a_n/2,\infty)\cap J_n} \psi_n(t)^{1/4}(1+|t|)^{-\alpha}/|x-t|\,dt$$

$$\sim a_n^{-\alpha}\int_{[a_n/2,\,x_{0n}]\setminus[x_{k(x)+3,n},\,x_{k(x)-3,n}]} \psi_n(t)^{1/4}/|x-t|\,dt$$

$$= a_n^{-\alpha}\int_{[a_n/2,\,x_{0n}]\setminus[x_{k(x)+3,n},\,x_{k(x)-3,n}]} \left(\max\{n^{-2/3}, 1-(|t|/a_n)\}\right)^{1/4}$$

$$/|x-t|\,dt$$

$$= a_n^{-\alpha}\int_{[1/2,\,x_{0n}/a_n]\setminus[x_{k(x)+3,n}/a_n,\,x_{k(x)-3,n}/a_n]} \left(\max\{n^{-2/3}, 1-s\}\right)^{1/4}$$

$$/|(x/a_n)-s|\,ds$$

$$= a_n^{-\alpha}\int_{K_n}\left(\max\{n^{-2/3},(1-(x/a_n))v\}\right)^{1/4}/|v-1|\,dv, \tag{2.17}$$

where we have used the substitution $1-s=(1-x/a_n)v$ and

$$K_n := \left[\frac{1-(x_{0n}/a_n)}{1-(x/a_n)}, \frac{1}{2(1-(x/a_n))}\right]\Bigg\backslash \left[\frac{1-(x_{k(x)+3,n}/a_n)}{1-(x/a_n)}, \frac{1-(x_{k(x)-3,n}/a_n)}{1-(x/a_n)}\right].$$

Now

$$\left|\frac{1-(x_{0n}/a_n)}{1-(x/a_n)}\right| = O\left(\frac{n^{-2/3}}{1-(x/a_n)}\right) = O(1/L) < 1/2$$

for L sufficiently large.

Then we can continue (2.17) as

$$\sim a_n^{-\alpha}\int_{K_n\cap(-\infty,1/2]}\left(\max\{n^{-2/3},(1-(x/a_n))v\}\right)^{1/4}/|v-1|\,dv$$

$$+ a_n^{-\alpha}\int_{K_n\cap[1/2,3/2]}\left(\max\{n^{-2/3},(1-(x/a_n))v\}\right)^{1/4}/|v-1|\,dv$$

$$+ a_n^{-\alpha}\int_{K_n\cap[3/2,\infty)}\left(\max\{n^{-2/3},(1-(x/a_n))v\}\right)^{1/4}|v-1|\,dv$$

$$=: I_{41}+I_{42}+I_{43}.$$

Note that $(x/a_n) \geq 3/4$ and so $1/2(1-(x/a_n)) \geq 2$. Now consider

I_{41}. Now $v \in K_n \cap (-\infty, 1/2] \Rightarrow |v-1| \sim 1$. So

$$I_{41} \sim a_n^{-\alpha} \int_{K_n \cap (-\infty, 1/2]} \left(\max\{n^{-2/3}, (1-(x/a_n))v\}\right)^{1/4} dv$$

$$\sim a_n^{-\alpha}\left[\int_0^{|1-(x_{0n}/a_n)|/(1-(x/a_n))} n^{-1/6}\, dv\right.$$

$$\left. + \int_{|1-(x_{0n}/a_n)|/(1-(x/a_n))}^{1/2} \left[(1-(x/a_n))v\right]^{1/4} dv\right]$$

$$\sim a_n^{-\alpha} n^{-1/6} \left|\frac{1-(x_{0n}/a_n)}{1-(x/a_n)}\right| + C_7 a_n^{-\alpha}(1-(x/a_n))^{1/4}.$$

But now

$$a_n^{-\alpha} n^{-1/6} \left|\frac{1-(x_{0n}/a_n)}{1-(x/a_n)}\right| \leq C_8 a_n^{-\alpha}(1-(x/a_n))^{1/4}.$$

Thus

$$I_{41} \sim a_n^{-\alpha}(1-(x/a_n))^{1/4}.$$

I_{42}. We have $(1-(x/a_n))v \sim 1-(x/a_n)$, and so

$$I_{42} \sim a_n^{-\alpha}\psi_n(x)^{1/4} \int_{K_n \cap [1/2, 3/4]} 1/|v-1|\, dv$$

$$= a_n^{-\alpha}\psi_n(x)^{1/4} \int_{[1/2, 3/2] \setminus \left[\frac{1-(x_{k(x)+3,n}/a_n)}{1-(x/a_n)}, \frac{1-(x_{k(x)-3,n}/a_n)}{1-(x/a_n)}\right]} 1/|v-1|\, dv$$

$$= a_n^{-\alpha}\psi_n(x)^{1/4}\left[\int_{\left[1/2, \frac{1-(x_{k(x)+3,n}/a_n)}{1-(x/a_n)}\right]}\right.$$

$$\left. + \int_{\left[\frac{1-(x_{k(x)-3,n}/a_n)}{1-(x/a_n)}, 3/2\right]}\right] 1/|v-1|\, dv$$

$$= a_n^{-\alpha}\psi_n(x)^{1/4}\left[\log\left[\frac{1-(x/a_n)}{2(x_{k(x)-3,n}-x)/a_n}\right]\right.$$

$$\left. + \log\left[\frac{1-(x/a_n)}{2(x-x_{k(x)+3,n})/a_n}\right]\right].$$

Observe that

$$\left|\frac{1 - x_{k\pm 3,n}}{1 - (x/a_n)} - 1\right| = \left|\frac{(x - x_{k\pm 3,n})/a_n}{1 - (x/a_n)}\right| \sim \frac{(x_{k+1,n} - x_{k+3,n})/a_n}{1 - (x/a_n)}$$

$$\sim \frac{\psi_n(x)^{-1/2}}{n(1 - (x/a_n))} \sim \frac{(1 - (x/a_n))^{-3/2}}{n}$$

as $[1 - (x/a_n)] \geq Ln^{-2/3}$ for $x \in (x_{k+1,n}, x_{k-1,n})$ and

$$[1 - (x_{k\pm 2,n}/a_n)] \sim [1 - (x/a_n)].$$

It follows that

$$I_{42} \sim a_n^{-\alpha}\psi_n(x)^{1/4}\log\left\{n[1 - (x/a_n)]^{3/2}\right\}$$

$$\sim a_n^{-\alpha}\psi_n(x)^{1/4}\log\left[2n^{2/3}\psi_n(x)\right].$$

Now $n^{2/3}\psi_n(x) \geq L$. Thus $I_{42} > I_{41}$ and so

$$I_{41} + I_{42} \sim a_n^{-\alpha}\psi_n(x)^{1/4}\log\left[2n^{2/3}\psi_n(x)\right].$$

Furthermore,

$$I_{43} = a_n^{-\alpha}\int_{K_n\cap[3/2,\infty)}[1 - (x/a_n)]^{1/4}v^{1/4}/|v - 1|\,dv$$

$$= a_n^{-\alpha}[1 - (x/a_n)]^{1/4}\int_{K_n\cap[3/2,\infty)} v^{-3/4}\,dv$$

as $v \in K_n \cap [3/2,\infty) \Rightarrow v - 1 \sim v$. Hence

$$I_{43} \sim a_n^{-\alpha}\psi_n(x)^{1/4}v^{1/4}\Big|_{3/4}^{\frac{1}{2[1 - (x/a_n)]}}$$

$$\sim a_n^{-\alpha}\psi_n(x)^{1/4}\left[\psi_n(x)^{-1/4} - (3/4)^{1/4}\right]$$

$$\sim a_n^{-\alpha},$$

since $\psi_n(x)^{-1/4} - 1 \sim \psi_n(x)^{-1/4}$. Thus

$$I_4 = I_{41} + I_{42} + I_{43}$$

$$\sim a_n^{-\alpha}\left\{[1 - (x/a_n)]^{1/4} + \psi_n(x)^{1/4}\log\left[2n^{2/3}\psi_n(x)\right] + 1\right\}$$

$$\sim a_n^{-\alpha}\psi_n(x)^{1/4}\log\left[2n^{2/3}\psi_n(x)\right].$$

Therefore

$$\begin{aligned} I &= I_3 + I_4 \\ &\sim a_n^{-\hat{\alpha}} \log^* n + a_n^{-\alpha} \psi_n(x)^{1/4} \log\left[2n^{2/3} \psi_n(x)\right]. \quad \blacksquare \end{aligned}$$

Now from Lemma 2.6 and Eqs. (2.15) and (2.16) we obtain

LEMMA 2.7. *For $x \in [(3/4)a_n, a_n(1 - Ln^{-2/3})]$, $L > 0$, large enough,*

$$\hat{\Sigma}_3(x) \sim \sqrt{a_n} |p_n W|(x) \left\{ a^{-\hat{\alpha}} \log^* n + a_n^{-\alpha} \psi_n(x)^{1/4} \log\left[2n^{2/3} \psi_n(x)\right] \right\}.$$

Remark. Observe that for $|x| \le (3/4)a_n$, $\psi_n(x) \sim 1$ and $n^{2/3}\psi_n(x) \sim n^{2/3}$. So

$$\log\left[2n^{2/3}\psi_n(x)\right] \sim \log n.$$

Thus for $|x| \le (3/4)a_n$, we can recast Lemma 2.5 as

$$\begin{aligned} \hat{\Sigma}_3(x) \sim \sqrt{a_n} |p_n W|(x) \Big\{ &(1 + |x|)^{-\alpha} \psi_n(x)^{1/4} \log\left[2n^{2/3}\psi_n(x)\right] \\ &+ (1 + |x|)^{\hat{\alpha}} \log^* n \Big\}. \end{aligned}$$

Thus we have

LEMMA 2.8. *Let $|x| \le a_n(1 - Ln^{-2/3})$. Then*

$$\begin{aligned} \hat{\Sigma}_3(x) \sim \sqrt{a_n} |p_n W|(x) \Big\{ &(1 + |x|)^{-\alpha} \psi_n(x)^{1/4} \log\left[2n^{2/3}\psi_n(x)\right] \\ &+ (1 + |x|)^{-\hat{\alpha}} \log^* n \Big\}. \end{aligned}$$

LEMMA 2.9. *Let $|1 - (x/a_n)| \le Ln^{-2/3}$, $L > 0$ large enough. Then*

$$\begin{aligned} \hat{\Sigma}_3(x) \sim \sqrt{a_n} |p_n W|(x) \Big\{ &(1 + |x|)^{-\alpha} \psi_n(x)^{1/4} \log\left[2n^{2/3}\psi_n(x)\right] \\ &+ (1 + |x|)^{-\hat{\alpha}} \log^* n \Big\}. \end{aligned}$$

Proof. Here we write

$$\begin{aligned} I &\sim \int_{J_n \cap [-a_n/2,\, a_n/2]} \psi_n(t)^{1/4} (1 + |t|)^{-\alpha} / |x - t| \, dt \\ &\quad + \int_{J_n \cap [a_n/2,\, 2a_n]} \psi_n(t)^{1/4} (1 + |t|)^{-\alpha} / |x - t| \, dt \\ &=: I_5 + I_6 \end{aligned}$$

since all the zeros of $p_n(x)$ are inside $[-2a_n, 2a_n]$.

Now for $t \in J_n \cap [-a_n/2, a_n/2]$, $\psi_n(t) \sim 1$ and $|x - t| \sim a_n$. So

$$I_5 \sim a_n^{-1} \int_{J_n \cap [-a_n/2, a_n/2]} (1 + |t|)^{-\alpha}\, dt$$
$$\sim a_n^{-\hat{\alpha}} \log {}^* n.$$

Next consider I_6. For this range we have $t \sim a_n$. Thus

$$I_6 \sim a_n^{-\alpha} \int_{J_n \cap [a_n/2, 2a_n]} \psi_n(t)^{1/4}/|x - t|\, dt$$
$$= a_n^{-\alpha} \int_{J_n \cap [a_n/2, 2a_n] \cap \{t: |(t/a_n) - 1| \le 2Ln^{-2/3}\}} \psi_n(t)^{1/4}/|x - t|\, dt$$
$$+ a_n^{-\alpha} \int_{J_n \cap [a_n/2, 2a_n] \cap \{t: |(t/a_n) - 1| \ge 2Ln^{-2/3}\}} \psi_n(t)^{1/4}/|x - t|\, dt$$
$$=: I_{61} + I_{62}.$$

Here for large enough n

$$I_{61} \sim a_n^{-\alpha} n^{-1/6} \int_{J_n \cap \{t: |(t/a_n) - 1| \le 2Ln^{-2/3}\}} 1/|x - t|\, dt$$
$$= a_n^{-\alpha} n^{-1/6} \int_{J_n/a_n \cap \{s: |s - 1| \le 2Ln^{-2/3}\}} 1/\left|(x/a_n) - s\right| ds$$
$$\le C_9 a_n^{-\alpha} n^{-1/6} \log n \le C_9 a_n^{-\alpha} \log n.$$

Next

$$\left|(t/a_n) - 1\right| \ge 2Ln^{-2/3}$$
$$\Rightarrow |x - t| = a_n\left|[1 - (t/a_n)] - [1 - (x/a_n)]\right| \ge a_n|1 - (t/a_n)|/2$$

as $|1 - (x/a_n)| \le Ln^{-2/3} \le (1/2)|1 - (t/a_n)|$. Hence

$$I_{62} \le C_{10} a^{-\alpha-1} \int_{J_n \cap [a_n/2, 2a_n] \setminus \{t: |(t/a_n) - 1| \ge 2Ln^{-2/3}\}} \left(\max\{n^{-2/3}, 1 - (t/a_n)\}\right)^{1/4} / |1 - (t/a_n)|\, dt$$
$$\le C_{11} a_n^{-\alpha-1} \int_{J_n} |1 - (t/a_n)|^{-3/4}\, dt$$
$$= C_{11} a_n^{-\alpha} \int_{J_n/a_n} |1 - s|^{-3/4}\, ds$$
$$\sim a_n^{-\alpha},$$

as $(1-s)^{-3/4}$ is integrable. Hence

$$\begin{aligned} I &\sim a_n^{-\hat{\alpha}} \log^* n + I_{61} + I_{62} \\ &\sim a_n^{-\hat{\alpha}} \log^* n, \end{aligned}$$

and thus

$$\begin{aligned} \hat{\Sigma}_3(x) &\sim \sqrt{a_n}\, |p_n W|(x) a_n^{-\hat{\alpha}} \log^* n \\ &\sim \sqrt{a_n}\, |p_n W|(x) \Big\{ (1+|x|)^{-\alpha} \psi_n(x)^{1/4} \log\big[2n^{2/3}\psi_n(x)\big] \\ &\qquad + (1+|x|)^{-\hat{\alpha}} \log^* n \Big\} \end{aligned}$$

since $\psi_n(x)^{1/4} \log[2n^{2/3}\psi_n(x)] = O(n^{-1/6}\log n) = o(1)$. ∎

Consequently Lemma 2.8 and Lemma 2.9 yield

LEMMA 2.10. *For $|x| \le a_n(1 + Ln^{-2/3})$, $L > 0$, large enough,*

$$\begin{aligned} \hat{\Sigma}_3(x) \sim \sqrt{a_n}\, |p_n W|(x) \Big\{ &(1+|x|)^{-\alpha} \psi_n(x)^{1/4} \log\big[2n^{2/3}\psi_n(x)\big] \\ &+ (1+|x|)^{-\hat{\alpha}} \log^* n \Big\}. \end{aligned}$$

LEMMA 2.11. *Let $x \in [a_n(1 + Ln^{-2/3}), 2a_n]$, $L > 0$ large enough. Then*

$$\begin{aligned} \hat{\Sigma}_3(x) \sim \sqrt{a_n}\, |p_n W|(x) \Big\{ &(1+|x|)^{-\alpha} \psi_n(x)^{1/4} \log\big[2n^{2/3}\psi_n(x)\big] \\ &+ (1+|x|)^{-\hat{\alpha}} \log^* n \Big\}. \end{aligned}$$

Proof. If L is large enough, we have $|x_{k(x)+3,n}| \le a_n(1 + Ln^{-2/3})$. Then

$$\begin{aligned} I &:= \int_{J_n} \psi_n(t)^{1/4}(1+|t|)^{-\alpha}/|x-t|\, dt \\ &\sim \int_{[0,\, x_{0n}]} \psi_n(t)^{1/4}(1+|t|)^{-\alpha}/|x-t|\, dt \\ &= \int_{[0,\, a_n/2]} \psi_n(t)^{1/4}(1+|t|)^{-\alpha}/|x-t|\, dt \\ &\quad + \int_{[a_n/2,\, x_{0n}]} \psi_n(t)^{1/4}(1+|t|)^{-\alpha}/|x-t|\, dt \\ &=: I_7 + I_8. \end{aligned}$$

Now

$$I_7 \sim a_n^{-1} \int_0^{a_n/2} (1+t)^{-\alpha}\, dt$$
$$\sim a_n^{-\hat{\alpha}} \log^* n.$$

Now if L is large enough, we have for $t \in [a_n/2, x_{0n}]$, that

$$\begin{aligned} |x-t| &\geq x_{0n}(1+n^{-2/3}) - t \\ &\geq x_{0n} \max\{n^{-2/3}, 1-(t/x_{0n})\} \\ &\geq C_{12} a_n \max\{n^{-2/3}, 1-(t/a_n)\} \\ &= C_{12} a_n \psi_n(t) \end{aligned}$$

in view of (1.9) and (2.1). So

$$\begin{aligned} I_8 &\leq C_{13} a_n^{-\alpha} \int_{[a_n/2,\, x_{0n}]} \psi_n(t)^{1/4} / |x-t|\, dt \\ &\leq C_{14} a_n^{-\alpha-1} \int_{[a_n/2,\, x_{0n}]} \psi_n(t)^{-3/4}\, dt \\ &= C_{14} a_n^{-\alpha} \int_{[1/2,\, x_{0n}/a_n]} \left(\max\{n^{-2/3}, 1-s\}\right)^{-3/4} ds \\ &\leq C_{16} a_n^{-\alpha}. \end{aligned}$$

Thus

$$I \sim a_n^{-\hat{\alpha}} \log^* n.$$

Also in this case,

$$\psi_n(x)^{1/4} \log\left[2n^{2/3}\psi_n(x)\right] = O(n^{-1/6} \log n) = o(1).$$

Therefore

$$\hat{\Sigma}_3(x) \sim \sqrt{a_n}\, |p_n W|(x) \Big\{ (1+|x|)^{-\alpha} \psi_n(x)^{1/4} \log\left[2n^{2/3}\psi_n(x)\right] + (1+|x|)^{-\hat{\alpha}} \log^* n \Big\}. \quad \blacksquare$$

LEMMA 2.12. *There exists* n_0 *such that uniformly for* $n \geq n_0$ *and* $x \in [2a_n, \infty)$,

$$\Lambda_n(x) \sim \sqrt{a_n}\, |p_n W|(x) \frac{1}{|x|} a_n^{1-\hat{\alpha}} \log^* n.$$

Proof. Recall that all the zeros of $p_n(x)$ lie in $\{x\colon |x| \le a_n(1 + Ln^{-2/3})\}$, for some fixed large $L > 0$. So for n large enough,

$$|x - x_{kn}| \sim |x|, \qquad \text{uniformly for } 1 \le k \le n, \text{ and } x \ge 2a_n.$$

Then (2.5) gives uniformly for $x \ge 2a_n$,

$$\Lambda_n(x) \sim \sqrt{a_n}\,|p_n W|(x) \frac{1}{|x|} \sum_{k=1}^{n} \psi_n(x_{kn})^{-1/4} (1 + |x_{kn}|)^{-\alpha} \frac{a_n}{n}.$$

As in our estimate for $\hat{\Sigma}_3$, we deduce that

$$\begin{aligned} \Lambda_n(x) &\sim \sqrt{a_n}\,|p_n W|(x) \frac{1}{|x|} \int_{x_{n+1,n}}^{x_{0n}} \psi_n(t)^{1/4} (1 + |t|)^{-\alpha}\, dt \\ &\sim \sqrt{a_n}\,|p_n W|(x) \frac{1}{|x|} a_n^{1-\hat{\alpha}} \log^* n, \end{aligned}$$

exactly as in Lemma 2.6. ∎

3. Proof of the Result

Proof of Theorem 1.1. (a) For $|x| \le 2a_n$, (2.10), (2.14), and Lemmas 2.10, 2.11 give

$$\begin{aligned} \Lambda_n(x) &= \Sigma_1(x) + \Sigma_2(x) + \Sigma_3(x) \\ &\le C_1 (1 + |x|)^{-\alpha} \\ &\quad + C_1 \sqrt{a_n}\,|p_n W|(x) \psi_n(x)^{1/4} (1 + |x|)^{-\alpha} \\ &\quad + C_1 \sqrt{a_n}\,|p_n W|(x) \Big\{ (1 + |x|)^{-\alpha} \psi_n(x)^{1/4} \log\big[2n^{2/3} \psi_n(x)\big] \\ &\qquad\qquad + (1 + |x|)^{-\hat{\alpha}} \log^* n \Big\}. \end{aligned} \tag{3.1}$$

Since $2n^{2/3}\psi_n(x) \ge 1$, we see that the middle term of the sum may be omitted, and we obtain (1.10). Since (2.7), (2.8) show that

$$\sqrt{a_n}\,|p_n W|(x) \psi_n(x)^{1/4} \le C_2 \tag{3.2}$$

and

$$\sqrt{a_n}\,|p_n W|(x) \le C_2 n^{1/6} \tag{3.3}$$

we obtain also (1.11).

(b) For $|x| \le \sigma a_n$ the sums Σ_1 and Σ_2 are non-empty because of the spacing (2.2), if δ and M are suitably chosen. Then (2.12), (2.14), and Lemma 2.10 give (3.1) with $\le$ replaced by $\sim$. Moreover for this range,

$$\psi_n(x) \sim 1$$

and we obtain (1.12): We no longer need $\log^* n$ because if $\alpha = 1$, the $\log n$ is already present. Using our bounds (3.2) for $p_n(x)$ gives (1.13).

(c) This is Lemma 2.12. ∎

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